

SULIT



First Semester Examination
Academic Session 2018/2019

December 2018/January 2019

MAT363 - Statistical Inference
(Pentaabiran Statistik)

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN (7) pages of printed material before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH (7) muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions : Answer **FOUR (4)** questions.

[Arahan : Jawab **EMPAT (4)** soalan.]

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai].

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Question 1

- (a) Consider three events A , B and E such that: $P(A)=0.3$, $P(AB)=3P(ABE)$, $P(A \cup E)=0.5$, $P(AB^c)=0.03$, $P(BE)=0.1$ and $P(A^c B^c E^c)=0.48$.

- (i) Obtain $P(ABE)$
- (ii) Are events AB and AE independent?

- (b) If the joint probability density of X and Y is given by

$$f(x) = \begin{cases} 1+x & \text{for } -1 < x \leq 0 \\ 1-x & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

and $U = X$ and $V = X^2$, show that

- (i) $\text{cov}(U, V) = 0$
- (ii) U and V are dependent
- (c) The joint pdf of X and Y is given by $f(x, y) = \frac{1}{\sqrt{2\pi}} e^{-y} e^{-(x-y)^2/2}$, $0 < y < \infty$, $-\infty < x < \infty$. Compute the joint moment generating function of X and Y .

[25 marks]

Soalan 1

- (a) Pertimbangkan tiga peristiwa A , B dan E supaya: $P(A)=0.3$, $P(AB)=3P(ABE)$, $P(A \cup E)=0.5$, $P(AB^c)=0.03$, $P(BE)=0.1$ dan $P(A^c B^c E^c)=0.48$.

- (i) Dapatkan $P(ABE)$
- (ii) Adakah peristiwa AB dan AE tak bersandar?

- (b) Jika ketumpatan kebarangkalian tercantum X dan Y diberi oleh

$$f(x) = \begin{cases} 1+x & \text{untuk } -1 < x \leq 0 \\ 1-x & \text{untuk } 0 < x < 1 \\ 0 & \text{selainnya} \end{cases}$$

dan $U = X$ dan $V = X^2$, tunjukkan bahawa

- (i) $\text{cov}(U, V) = 0$
- (ii) U dan V adalah bersandar
- (c) Fungsi ketumpatan kebarangkalian tercantum X dan Y adalah diberi oleh $f(x, y) = \frac{1}{\sqrt{2\pi}} e^{-y} e^{-(x-y)^2/2}$, $0 < y < \infty$, $-\infty < x < \infty$. Terbitkan fungsi penjana momen tercantum X dan Y .

[25 markah]

Question 2

- (a) Let X_A be the indicator random variable for event A with probability $P(A)=0.8$. Let $\hat{P}(A)$ denote the relative frequency of event A in n independent trials.
- (i) Find $E[X_A]$ and $\text{var}[X_A]$
- (ii) Use the Chebyshev's inequality to find the confidence coefficient $1-\alpha$ such that $P\left\{\left|\hat{P}_{100}(A) - P(A)\right| \leq 0.1\right\} \geq 1-\alpha$
- (b) Consider two random variables X and Y with the joint probability density

$$f(x, y) = \begin{cases} 12xy(1-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

- (i) Find the probability density of $Z = XY^2$
- (ii) Find the joint probability density of $Z = XY^2$ and $U=Y$

[25 marks]

Soalan 2

(a) Biarkan X_A sebagai pembolehubah rawak bagi peristiwa A dengan kebarangkalian $P(A)=0.8$. Biarkan $\hat{P}(A)$ mewakili kekerapan relatif untuk peristiwa A dalam n percubaan tak bersandar.

(i) Cari $E[X_A]$ dan $\text{var}[X_A]$

(ii) Guna ketaksamaan Chebyshev untuk mencari pekali keyakinan $1-\alpha$ supaya $P\left\{\left|\hat{P}_{100}(A)-P(A)\right|\leq 0.1\right\}\geq 1-\alpha$

(b) Pertimbangkan dua pembolehubah rawak petunjuk X dan Y dengan ketumpatan kebarangkalian tercantum $f(x, y) = \begin{cases} 12xy(1-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{selainnya} \end{cases}$

(i) Cari ketumpatan kebarangkalian bersama bagi $Z = XY^2$

(ii) Cari ketumpatan kebarangkalian bersama bagi $Z = XY^2$ dan $U=Y$

[25 markah]

Question 3

(a) Let X_1, \dots, X_n be a random sample from $\exp(\lambda)$.

(i) Provide the maximum likelihood estimate for λ

(ii) Provide the method of moments estimate for λ

(b) Y_1, \dots, Y_n be a random sample of size n from the pdf $f_Y(y; \theta) = \frac{1}{(r-1)!\theta^r} y^{r-1} e^{-y/\theta}$; $y > 0$. It is given that $E(Y) = r\theta$ and $\text{var}(Y) = r\theta^2$.

(i) Is $\hat{\theta} = \frac{1}{r} \bar{Y}$ an unbiased estimator for θ ?

(ii) Show that $\hat{\theta} = \frac{1}{r} \bar{Y}$ is a minimum variance estimator for θ .

[25marks]

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Soalan 3

- (a) Biarkan X_1, \dots, X_n sebagai sampel rawak dari $\text{eks}(\lambda)$.
- (i) Berikan anggaran kebolehjadian maksimum bagi λ
- (ii) Berikan anggaran kaedah momen bagi λ
- (b) Y_1, \dots, Y_n merupakan sampel rawak bersaiz n dari fungsi ketumpatan kebarangkalian $f_Y(y; \theta) = \frac{1}{(r-1)!\theta^r} y^{r-1} e^{-y/\theta}$; $y > 0$. Adalah diberi bahawa $E(Y) = r\theta$ dan $\text{var}(Y) = r\theta^2$.
- (i) Adakah $\hat{\theta} = \frac{1}{r} \bar{Y}$ penganggar saksama bagi θ ?
- (ii) Tunjukkan bahawa $\hat{\theta} = \frac{1}{r} \bar{Y}$ merupakan penganggar varians minimum bagi θ .

[25 markah]

Question 4

- (a) Let X be a single observation from a density function $f(x, \theta) = \theta x^{\theta-1}$, $0 < x < 1$ and $\theta > 0$. If $-\theta \log X$ is a pivotal quantity, use it to construct a confidence interval for θ with the level of confidence $1 - \alpha$.
- (b) Let X_1, \dots, X_n be a random sample from the normal distribution $N(\mu, 196)$.
- (i) To test $H_0: \mu = 56$ against $H_1: \mu \neq 56$, what is the critical region of size $\alpha = 0.05$ specified by the likelihood ratio test criterion? $z_{0.025} = 1.96$
- (ii) If a sample of size $n=100$ yielded $\bar{x} = 56.2$, is H_0 accepted?
- (iii) Find the sample size if we are 95% confident that the maximum error of estimate of μ is 1.5.
- (c) Let X_1 and X_2 be a random sample of size 2 from the Poisson distribution $f(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$. Show that $T = X_1 + X_2$ is a sufficient statistic.

[25 marks]

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Soalan 4

- (a) Biarkan X sebagai suatu cerapan tunggal dari fungsi ketumpatan $f(x, \theta) = \theta x^{\theta-1}$, $0 < x < 1$ dan $\theta > 0$. Jika $-\theta \log X$ ialah kuantiti pangsaan, gunakannya untuk membina satu selang keyakinan bagi θ dengan aras keyakinan $1 - \alpha$.
- (b) Biarkan X_1, \dots, X_n sebagai satu sampel rawak dari taburan normal $N(\mu, 196)$.
- (i) Untuk menguji $H_0: \mu = 56$ lawan $H_1: \mu \neq 56$, apakah rantau genting bersaiz $\alpha = 0.05$ yang dispesifikasikan oleh kriteria ujian nisbah kebolehdian? ($z_{0.025} = 1.96$).
- (ii) Jika sampel bersaiz $n=100$ memberi $\bar{x} = 56.2$, adakah H_0 diterima?
- (iii) Cari saiz sampel jika kita adalah 95% yakin bahawa ralat maksimum bagi anggaran μ ialah 1.5.
- (c) Biarkan X_1 and X_2 merupakan pembolehubah rawak bersaiz 2 dari taburan Poisson distribution $f(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$. Tunjukkan bahawa $T = X_1 + X_2$ adalah statistik mencukupi.

[25 markah]

Appendix

Discrete Distributions	
Bernoulli $0 < p < 1$	$f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$ $M(t) = 1 - p + pe^t$ $\mu = p, \quad \sigma^2 = p(1-p)$
Binomial $b(n, p)$ $0 < p < 1$	$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n$ $M(t) = (1-p + pe^t)^n$ $\mu = np, \quad \sigma^2 = np(1-p)$
Geometric $0 < p < 1$	$f(x) = (1-p)^{x-1}p, \quad x = 1, 2, 3, \dots$ $M(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\ln(1-p)$ $\mu = \frac{1}{p}, \quad \sigma^2 = \frac{1-p}{p^2}$
Hypergeometric $N_1 > 0, N_2 > 0$ $N = N_1 + N_2$	$f(x) = \frac{\binom{N_1}{x} \binom{N_2}{n-x}}{\binom{N}{n}}, \quad x \leq n, x \leq N_1, n - x \leq N_2$ $\mu = n \left(\frac{N_1}{N} \right), \quad \sigma^2 = n \left(\frac{N_1}{N} \right) \left(\frac{N_2}{N} \right) \left(\frac{N-n}{N-1} \right)$
Negative Binomial $0 < p < 1$ $r = 1, 2, 3, \dots$	$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, r+2, \dots$ $M(t) = \frac{(pe^t)^r}{[1 - (1-p)e^t]^r}, \quad t < -\ln(1-p)$ $\mu = r \left(\frac{1}{p} \right), \quad \sigma^2 = \frac{r(1-p)}{p^2}$
Poisson $0 < \lambda$	$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots$ $M(t) = e^{\lambda(e^t-1)}$ $\mu = \lambda, \quad \sigma^2 = \lambda$
Uniform $m > 0$	$f(x) = \frac{1}{m}, \quad x = 1, 2, \dots, m$ $\mu = \frac{m+1}{2}, \quad \sigma^2 = \frac{m^2-1}{12}$
Continuous Distributions	
Beta $0 < \alpha$ $0 < \beta$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$ $\mu = \frac{\alpha}{\alpha+\beta}, \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$
Chi-square $\chi^2(r)$ $r = 1, 2, \dots$	$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad 0 \leq x < \infty$ $M(t) = \frac{1}{(1-2t)^{r/2}}, \quad t < \frac{1}{2}$ $\mu = r, \quad \sigma^2 = 2r$
Exponential $0 < \theta$	$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 \leq x < \infty$ $M(t) = \frac{1}{1-\theta t}, \quad t < \frac{1}{\theta}$ $\mu = \theta, \quad \sigma^2 = \theta^2$
Gamma $0 < \alpha$ $0 < \theta$	$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 \leq x < \infty$ $M(t) = \frac{1}{(1-\theta t)^\alpha}, \quad t < \frac{1}{\theta}$ $\mu = \alpha\theta, \quad \sigma^2 = \alpha\theta^2$
Normal $N(\mu, \sigma^2)$ $-\infty < \mu < \infty$ $0 < \sigma$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$ $M(t) = e^{i\mu t + \sigma^2 t^2/2}$ $E(X) = \mu, \quad \text{Var}(X) = \sigma^2$
Uniform $U(a, b)$ $-\infty < a < b < \infty$	$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$ $M(t) = \frac{e^{bt} - e^{at}}{t(b-a)}, \quad t \neq 0; \quad M(0) = 1$ $\mu = \frac{a+b}{2}, \quad \sigma^2 = \frac{(b-a)^2}{12}$

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